Lec 3: 09/04/2013

Friedmann-Robertson-Walker Universe (Cont'd):

Let us use the Friedmann equations and find time evolution of the scale factor $a(t)$ in some important cases:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \quad (\text{I})$$

(We use $c=1$ from now on, unless it is mentioned otherwise)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \quad (\text{II})$$

In order to find $a(t)$, we need to know the dependence of $\rho_{(t)}$ on $a_{(t)}$. This depends on the relation between P and ρ for a fluid, which is called the equation of state:

$$P = \omega \rho$$

Some important examples are $\omega = 0$ ($P=0$) for non-relativistic matter, $\omega = \frac{1}{3}$ ($P = \frac{1}{3}\rho$) for relativistic particles, and $\omega = -1$ ($P = -\rho$) for vacuum energy (or cosmological constant).

From the equation of state and the first law of thermodynamics we can find the change in ρ as a result of expansion.

Recall the first law of thermodynamics:

$dE = T\,dS - P\,dV \qquad E = \rho V \Rightarrow dE = \rho\, dV + V\, d\rho$

$$\frac{dE}{dt} = \rho \frac{dV}{dt} + V \frac{d\rho}{dt} = -P \frac{dV}{dt}$$

In a homogeneous and isotropic universe temperature (T) is the same everywhere. There can therefore be no change in the entropy (which would require heat transfer, hence temperature difference). In consequence, we have:

$\rho \dot{V} + V \dot{\rho} = -P \dot{V} \Rightarrow \dot{\rho} V = -(P+\rho)\dot{V}$

Since the volume is $V_{(t)} \propto a^3_{(t)}$, then $\dot{V}_{(t)} \propto 3a^2_{(t)}\dot{a}_{(t)}$.

Thus:

$$\dot{\rho} = -(P+\rho)\frac{\dot{V}}{V} = -3\frac{\dot{a}(t)}{a(t)}(P+\rho) = -3H(P+\rho)$$

From the equation of state, we find:

$$\dot{\rho} = -3H(\omega\rho + \rho) = -3H(1+\omega)\rho \Rightarrow \frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a} \Rightarrow$$

$$\rho_{(t)} \propto a_{(t)}^{-3(1+\omega)} \qquad (\mathrm{III})$$

Equation (III) can now be used along with the first Friedmann equation to obtain $a(t)$:

1) Matter-dominated (MD) universe ($\omega = 0$). In this case, we have

$$\rho \propto a^{-3(1+\omega)} = a^{-3} \qquad \left(\rho = \rho_0 \frac{a_0^3}{a^3},\ \rho_0 \text{ being the density at } a_0\right)$$

Thus;

initial values of ρ and a at $t = t_0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_0}{3}\left(\frac{a_0}{a}\right)^3 - \frac{K}{a^2}$$

For simplicity, we consider a flat universe ($K=0$). This results in

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0 a_0^3}{3a^3} \Rightarrow \frac{da}{dt} = \left(\frac{8\pi G\rho_0 a_0^3}{3}\right)^{\frac{1}{2}} \frac{1}{a^{\frac{1}{2}}} \Rightarrow a^{\frac{1}{2}}\, da =$$

$$\left(\frac{8\pi G\rho_0 a_0^3}{3}\right)^{\frac{1}{2}} dt \Rightarrow a(t) \propto t^{\frac{2}{3}} \qquad (\text{IV})$$

2) Radiation-dominated (RD) universe ($\omega = \frac{1}{3}$). In this case, we have

$$\rho \propto a^{-3(1+\omega)} = a^{-4} \qquad \rho = \rho_0\left(\frac{a_0}{a}\right)^4$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_0}{3}\left(\frac{a_0}{a}\right)^4 \qquad (k=0 \text{ assumed})$$

Thus;

$$\frac{da}{dt} = \left(\frac{8\pi G\rho_0 a_0^4}{3}\right)^{\frac{1}{2}} \frac{1}{a} \Rightarrow a\,da = \left(\frac{8\pi G\rho_0 a_0^4}{3}\right)^{\frac{1}{2}} dt \Rightarrow$$

$a_{(t)} \propto t^{\frac{1}{2}}$ (V)

3) Universe dominated by cosmological constant ($\omega = -1$). In this case, we have:

"Einstein's Cosmological Constant:" $\Lambda \equiv 8\pi G \rho_\Lambda$

$\rho \propto a^{-3(1+\omega)} \Rightarrow \rho = \rho_0 \equiv \rho_\Lambda$ (ρ_Λ being constant)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_\Lambda}{3} \Rightarrow \frac{da}{a} = \left(\frac{8\pi G \rho_\Lambda}{3}\right)^{\frac{1}{2}} dt \Rightarrow d(\ln a) = \left(\frac{8\pi G \rho_\Lambda}{3}\right)^{\frac{1}{2}} dt$$

$$\Rightarrow a \propto \exp(H_0 t) \qquad H_0 \equiv \left(\frac{8\pi G \rho_\Lambda}{3}\right)^{\frac{1}{2}} \qquad \text{(VI)}$$

It is seen from above that $\rho_r \propto a^{-4}$, $\rho_m \propto a^{-3}$, $\rho_\Lambda = $ Const.. This implies that in an expanding universe the contribution of radiation to the total energy density decrease more rapidly as that of matter. As a result, matter becomes the dominant component after some time. Also, the contribution of matter decreases much more rapidly than that of the cosmological constant. Hence, no matter how small ρ_Λ is initiall, it will dominate the total energy density at some point.

To put things in perspective, from what we know about our universe, transition from RD to MD occured at $t \sim 50,000$ yr after the big bang. Transition from MD to dominance of cosmological constant (accelerat expansion era) happened at $t \sim 10^{10}$ yr. In this phase, the scale factor grows exponentially $a(t) \propto \exp(H_0 t)$, where $H_0 \simeq 67$ km s^{-1} Mpc^{-1} is inferred from fitting the observational data with the standard cosmological model.

It can be shown that for $\omega > -\frac{1}{3}$, we have $a(t) \propto t^{\alpha}$ where $\alpha < 1$. On the other hand, for $-1 < \omega \leq -\frac{1}{3}$, we find $a(t) \propto t^{\beta}$ where $\beta \geq 1$. As we saw, for $\omega = -1$ we have $a(t) \propto e^{-H_0 t}$. This implies that $\ddot{a} < 0$ for $\omega > -\frac{1}{3}$, $\ddot{a} = 0$ for $\omega = -\frac{1}{3}$, and $\ddot{a} > 0$ for $\omega < -\frac{1}{3}$. It is in line with the second Friedmann equation that gives rise to $\ddot{a} < 0$ for $P > -\frac{1}{3}\rho$

Before moving forward, let us pause for a moment to discuss some misconceptions and confusions that commonly arise in the context of an expanding universe:

(1) Apparent contradiction between expansion and special relativity. To elucidate, consider comoving coordinates (r, θ, ϕ) in a FRW expanding universe. These coordinates are called comoving because if an object is at a point with coordinates (r, θ, ϕ) at rest initially, it will always stay at the same point. This can be understood from the fact that in a homogeneous and isotropic universe there is no preferred direction, and hence an object at rest cannot start moving. Consider two points with comoving coordinates (r_1, θ_1, ϕ_1) and (r_2, θ_2, ϕ_2). The physical distance between these points is $a(t)\,\Delta r$, where $\Delta r = r_2 - r_1$ (assuming $r_2 > r_1$). The rate at which the physical distance between two

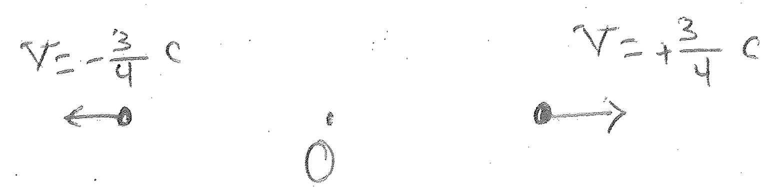
points grows is $\dot{a}_{(t)}\,\Delta r$. In an expanding universe $\dot{a}_{(t)}$ is an increasing function of t. It can therefore happen that $\dot{a}_{(t)}\,\Delta r$ exceed the speed of light c. This is very easy to see in a universe dominated by cosmological constants for which $a_{(t)} \propto \exp(H_0 t)$. In this case:

$\dot{a}_{(t)}\,\Delta r = H_0 \exp(H_0 t)\,\Delta r$

Thus, no matter how small Δr is, at some moment of time $\dot{a}_{(t)}\,\Delta r > c$. The question/confusion is whether such a "superluminal expansion" will be in contradiction to special relativity.

The answer is no. The important point is that two observers at points $(r_1, \theta_1, \varphi_1)$ and $(r_2, \theta_1, \varphi_1)$ cannot communicate through a physical signal that travels at a speed faster than c. This is what special relativity tells us, and remains valid in an expanding univer

Otherwise, the physical distance between two objects may increase at a speed larger than c without violating causality. As a simple example that illustrates this, consider two objects that move at a speed $\frac{3}{4}c$ in opposite direction:



According to an observer standing at point O, the distance between these objects grows at a speed $\frac{3}{2}c$. This, however, does not contradict special relativity because it does not correspond to propagation of any physical signal at superluminal velocities.

(2) Whether the distance between any two objects expands, or even any object itself expands. The question is which systems are undergoing expansion in the Universe. In fact

we can only notice and measure expansion if there are rulers that we can use for measuring distances that themselves have fixed lengths. If every object expands in the same way, there will be no way to find out about expansion. In such a case, expansion would have no physical effect.

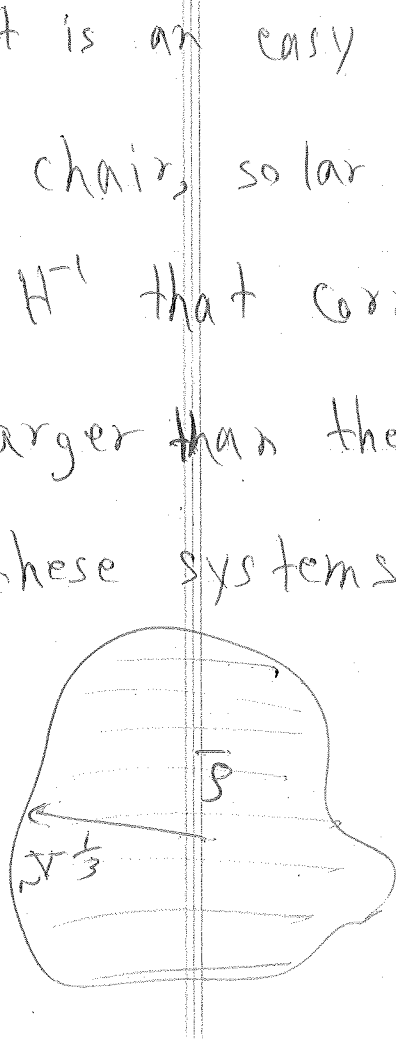
Systems that consist of parts bound to each other are not subject to expansion. This can include gravitationally bound systems (galaxies, clusters, etc) or those bound due to the forces from electromagetism (daily objects).

A quantitative criterion to find whether a system is expanding can be given as follows. Consider a region of volume V and calculate the average density $\bar{\rho}$ within this region. Inserting $\bar{\rho}$ in the first Friedmann equation, one finds an expansion rate

$$H^2 = \frac{8\pi G}{3}\bar{\rho} \Rightarrow H = \left(\frac{8\pi G}{3}\bar{\rho}\right)^{\frac{1}{2}}$$

We note that H has dimension (time)$^{-1}$. One can find a distance scale cH^{-1}. If $cH^{-1} < V^{\frac{1}{3}}$, then the system will expand at a rate $\simeq H$. If $cH^{-1} > V^{\frac{1}{3}}$, then there will be no expansion. A qualitative way to understand this is that cH^{-1} represents the physical distance that light can travel within a time interval H^{-1}. If $cH^{-1} > V^{\frac{1}{3}}$, then forces acting on different parts of the system can bind these parts together, and hence no expansion.

It is an easy exercise to show that for objects like a chair, solar system, milky way, etc the distance scale cH^{-1} that corresponds to the average density is much larger than the size of the system. This explains why these systems are bound by the relevant forces.



$$c\left(\frac{8\pi G\bar{\rho}}{3}\right)^{\frac{1}{2}} < V^{\frac{1}{3}} \longrightarrow \text{expansion}$$

$$c\left(\frac{8\pi G\bar{\rho}}{3}\right)^{\frac{1}{2}} > V^{\frac{1}{3}} \longrightarrow \text{bound}$$